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Source: *The Journal of the Operational Research Society*, SEPTEMBER 2015, Vol. 66, No. 9 (SEPTEMBER 2015), pp. 1454-1470

Published by: Palgrave Macmillan Journals on behalf of the Operational Research Society

Stable URL: https://www.jstor.org/stable/24505455

Journal of the Operational Research Society (2015) 66, 1454–1470

www.palgrave-journals.com/jors/

An optimization approach to resolving circular shareholding in large business groups

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Circular shareholding refers to a situation where a series of capital contributions made by companies in a family business group establish a chain of shareholdings. For example, a circular shareholding is formed when company A owns stock in company B, company B owns stock in company C, and company C owns stock in company A. In Korea, the practice of circular shareholding in large family-controlled business groups may give the principal families higher control over member firms and more opportunities to pursue their own interest at the expense of other shareholders. For this reason, the government of Korea has encouraged large conglomerates to gradually eliminate their circular shareholdings. However, there has been no research as to which shareholdings out of the complicated ownership structure should be cleared in order to resolve the issue of circular shareholding. In this paper, we propose optimization models to address the problem. Of the proposed integer programming models that can eliminate circular shareholding, one maximizes the sum of cash-flow rights while another maximizes the sum of voting rights. The proposed models have been applied to Korean family-controlled business groups, and the results are included herein. To the best knowledge of the authors, this research is the first study to apply optimization theory to the problem of resolving circular shareholding.

Journal of the Operational Research Society (2015) **66**(9), 1454–1470. doi:10.1057/jors.2014.114
Published online 26 November 2014

Keywords: circular shareholding; business group; optimization model; integer programming

1. Introduction

Like other Asian countries, Korea has dozens of large family business groups (or conglomerates) whose total sales account for more than half of its gross domestic product (Korea Fair Trade Commission, 2010). These conglomerates have played a crucial role in the growth of Korea's economy for many years; however, their rise has brought about other challenging issues, one of which is the concentration of economic power. Concerned with the immense influence of large conglomerates, the Korean government has introduced a series of policies in order to promote transparent corporate governance among domestic firms and create sound financial structures in its national economy.

Two Korean governmental policies are of interest to this study: first, restriction on cross-holdings, and second, a ceiling on total investment in member firms. A cross-holding is established when each of the two firms owns stock in the other firm. Since a cross-holding can inflate the fictitious capital of the two firms without any new financing from outside investors, it may be abused as a way to consolidate the principal family's control over the firms in its conglomerate. In fact, cross-holdings are prohibited by law in several countries including France, Germany, and Korea (La Porta *et al*, 1999). As of 2008, 41 Korean conglomerates involving 1044 firms are on record as complying with the policy (Korea Fair Trade Commission, 2010). On the other hand, the purpose of a ceiling on total investment is to limit the total amount of capital contributions one firm can make to other member firms in a business group. This policy aims at preventing a controlling shareholder from inflating his voting rights through a pyramidal ownership structure among member firms of a conglomerate. As of 2008, 14 Korean business groups involving 599 firms were judged to be in compliance with this policy (Korea Fair Trade Commission, 2010).

Despite the introduction of the two government policies mentioned above, principal conglomerate families in Korea still exercise excessive control over member firms within their business groups relative to their low ownership stake. The key instrument they have used to circumvent the prohibition on cross-holdings is through circular shareholding. Circular shareholding refers to a situation where a series of shareholdings by at least three member firms in a business group make up a chain of control. For instance, a circular shareholding is established when company A owns stock in company B, company B owns stock in company C, and company C owns stock in company A. Cross-holding can be seen as the simplest case of circular shareholding where only two companies hold each other's stock

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and create the simplest chain of ownership. Some literature does not distinguish between cross-holding and circular shareholding (La Porta *et al*, 1999; Claessens *et al*, 2000). However, while cross-holding is explicitly prohibited by law in Korea, circular shareholding is not against the law. Hence, this paper will distinguish cross-holding from circular shareholding. Just like cross-holding, circular shareholding among family-controlled firms can consolidate the voting rights of a controlling shareholder without any new inflow of capital from the outside. It may serve as a means for increasing the principal shareholder's control or affording more opportunities to pursue his private interest at the expense of minority investors. Thus, the Korean government has encouraged large business groups to gradually eliminate their circular shareholding. In addition, it has required large conglomerates to disclose complete ownership information and provide the public with a list of major circular shareholdings established by their member firms.

La Porta *et al* (1999) first examined the ownership structure of large publicly traded firms in different countries, including East Asian corporations. The study found that mechanisms, such as pyramidal ownership, cross-holdings, and circular shareholdings, helped principal owners of East Asian firms to take control of the firms without an absolute majority stock ownership. Claessens *et al* (2000) compared the separation between voting right and cash-flow right of family-owned companies across East Asian countries. Voting right refers to the ability to appoint a member of top management or influence important decisions that may need shareholder's approval, while cash-flow right refers to claims on dividends or cash payouts. Voting right is also referred to as control right, while cash-flow right is also referred to as ownership right. (The formal definition of voting right and cash-flow right will be given in Section 3.) The discrepancy between control and ownership is enlarged by a variety of factors, including pyramidal ownership and circular shareholdings. According to Claessens *et al* (2000), there is a significant divergence between the two rights in Korean firms. At the same time, other research has demonstrated that a high divergence between cash-flow right and voting right incentivizes a controlling shareholder to expropriate firm resources, leading to lower firm profitability and lower stock returns (Joh, 2003; Lemmon and Lins, 2003; Baek *et al*, 2004). Gompers *et al* (2003) and Bauer *et al* (2004) also found that corporate governance, that is, the strength of shareholder's right in comparison to manager's power, has a positive relationship with stock returns and firm valuation. Recently, Almeida *et al* (2008) proposed new metrics in order to assess the extent of pyramidal ownership in family business groups. Although we focus on interest conflicts among the stakeholders of a business group, it is worth mentioning that there exist other theoretical models for explaining what leads business group to establish pyramiding and circular shareholding: for example, the efficiency model, which regards corporate form like business group as the outcome of a rational decision to maximize efficiency in underdeveloped or imperfect markets, and the political economy approach, which emphasizes the role of political connections or government policy in corporate governance. (Refer to Haggard *et al* (2003) for details.)

After research has demonstrated that circular shareholding leads to the higher divergence between cash-flow right and voting right, which may ultimately result in lower profitability and stock returns (Claessens *et al*, 2000; Joh, 2003; Lemmon and Lins, 2003; Baek *et al*, 2004), Lim (2006) and Park (2006) discussed governmental regulations against the practice of circular shareholding in Korea. However, no studies have yet addressed how to resolve circular shareholding, that is, which shareholdings in the ownership structure should be cleared to resolve circular shareholding. The decision might be apparent for a small business group consisting of no more than a few firms. Unfortunately, some large family-controlled business groups have dozens of member firms, which own stock in other member firms and are partly owned by other member firms. Moreover, the act of clearing capital investments between firms will significantly impact the cash-flow rights and voting rights a controlling shareholder holds in the member firms. When resolving circular shareholding, some controlling shareholders might seek to retain as much cash-flow rights as possible, and others might wish to maximize voting rights. Hence, systematic models need to be developed which help those business groups to resolve circular shareholding while optimizing voting rights or cash-flow rights.

In this paper, we propose mixed-integer optimization models, each of which has a different set of constraints and goal for resolving circular shareholding. The models are built on a network representation capturing ownership structure between shareholders and firms. In the network representation, which is called an ownership network, a shareholding is mapped to an arc, and a circular shareholding corresponds to a cycle. Before proposing the optimization models, we first analyse the mathematical properties of cash-flow right and voting right in an ownership network, which may include circular shareholdings. On the basis of these properties, optimization models are proposed. The first model determines the minimum number of arcs (shareholdings) to be deleted in order to eliminate all cycles (circular shareholdings) in an ownership network. The second model identifies shareholdings that need to be cleared in order to maximize cash-flow rights under a constraint of no circular shareholdings remaining. The third model maximizes voting rights while removing all circular shareholdings. The last model tries to combine the third model and the fourth model and maximize a weighted sum of cash-flow rights and voting rights. To the best knowledge of the authors, this study is the first to enquire into how to change a business group's ownership structure in order to eliminate circular shareholding from the perspective of optimization theory.

The paper is organized as follows: In the next section, we briefly discuss the problems caused by circular shareholding.

In Section 3, we represent the ownership structure of a business group as a network, and discuss how to incorporate the constraint of resolving circular shareholding into a mathematical formulation. In Section 4, the mathematical properties of cash-flow right and voting right are investigated, and then mixed-integer programming models are proposed. In Section 5, the models are applied to Korean family business groups and the results are discussed. Finally, we provide a conclusion and some remarks regarding directions for future research.

2. Circular shareholding

In this section, we introduce some simple examples that illustrate the problems caused by circular shareholding.

First, let us consider a conglomerate that consists of three member firms. A rectangle in Figure 1 shows the ownership structure of each firm—the amount of capital contributed by an investor and its name (shown in parenthesis). In Figure 1(a), controlling family A invests US$10 million of capital in firms B, C, and D, respectively, in return for stock. (Throughout the paper, stockholder's voting right is assumed to be proportionate to the capital they invest. That is, we assume a one-vote-one-share structure, with no preferred stock, as preferred stock does not carry voting right.) Holding 50% of each firm's outstanding shares, family A has control over firms B, C, and D. In contrast, another family E in Figure 1(b) invests $10 million in firm F, which in turn invests $10 million in firm G. Firm G invests $10 million in firm H. Family E can take direct control over firm F, which in turn can control firm G. Thus, family E has ultimate control over all three member firms even though its capital contribution is only one-third of that made by family A. This illustrates how a pyramidal ownership structure enables a founding family to strengthen its control with only a small capital contribution. This phenomenon becomes more amplified in Figure 1(c). Family K sets up firm L by investing $5 million in it, and becomes a controlling shareholder of firm L. Later, family K expands business by establishing firm M, half of whose total capital ($20 million) comes from firm L. Subsequently, newly established firm M invests $10 million in firm N. Finally, firm N invests $5 million in firm L, which completes the circular shareholding. As a controlling shareholder of firm L, family K can control firm M through firm L and also exercise ultimate control over firm N through firm M. Thus, family K exercises complete control over its three member firms with only a $5 million capital contribution—less than contributions made by the two founding families A and E.

In contrast to the degree of control, the three families differ in cash-flow rights. Family A can take 50% of the cash dividends from firms B, C, and D. In the case of family E, they can directly claim 50% of the cash dividend from firm F, which in turn claims 50% of the cash dividend from firm G. In other words, in addition to their 50% dividend claim from firm F, family E can collect 25% of the cash dividend from firm G and 12.5% of the cash dividend from firm H. Finally, compared with the other two families, family K receives only 25% of the cash dividend from firm L and much smaller percentages of the cash dividends from firms M and N. Of the three controlling families, family K has the biggest gap between its cash-flow rights and its degree of control. This is a common phenomenon caused by circular shareholding.

Circular shareholding is common in many family-controlled Korean conglomerates. As of 2007, 18 business groups, including Samsung and Hyundai Motors, have established circular shareholdings. Despite a ceiling on total investment in member firms, many business groups still practice their circular shareholding. While further governmental measures are under consideration, there has been continuous pressure on Korean conglomerates to resolve their circular shareholding. However, no research has addressed how to eliminate all circular shareholdings from the perspective of optimization theory.

3. Network representation

In this section, we introduce the network representation of shareholdings between firms, and then discuss how to formulate the constraints of eliminating all circular shareholdings into mathematical terms.

Consider a business group that has one controlling family and five firms. Table 1 shows shares owned by each firm or the controlling family.

The shareholding structure of the business group above can be represented as a network, as shown in Figure 2. (A network such as shown in Figure 1 will be called an *ownership network.*) Each node in the network corresponds to a firm or a controlling family. If firm (or controlling family) i owns some shares issued by firm j, then node i is connected with node j by arc (i, j) going from node i to j. Also, arc (i, j) is associated with a percentage of ownership firm i holds in firm j. For example, in Figure 2, arc $(1, 2)$ represents a 40% ownership stake of controlling family 1 in firm 2.

Before further discussion, we introduce some basic terms used in graph theory. Let N and E denote the set of nodes and the set of arcs where n is the number of nodes in N. A sequence of distinct nodes, $(i=i_1, i_2, \ldots, j=i_r)$, with $r \geqslant 2$ is called a *path* if $(i_k, i_{k+1}) \in E$ for all $1 \leqslant k \leqslant r-1$. A sequence of nodes, $(i_1, i_2, \ldots, i_r)$, with $r \geqslant 3$ is called a *cycle* if $(i_1, i_2, \ldots, i_{r-1})$ is a path and $(i_{r-1}, i_r) = (i_{r-1}, i_1) \in E$. If a firm or family can exercise some level of control over another firm, there must exist at least one path between them. For example, controlling family 1 can control firm 2 directly as arc $(1, 2)$ indicates. Also, controlling family 1 can exercise control over firm 2 indirectly through a path $(1, 3, 5, 4, 2)$. Moreover, a circular shareholding forms a cycle in an ownership network. There are two cycles in Figure 2, $(3, 5, 4, 3)$ and $(2, 3, 5, 4, 2)$, which indicates that Table 1 includes two circular shareholdings.

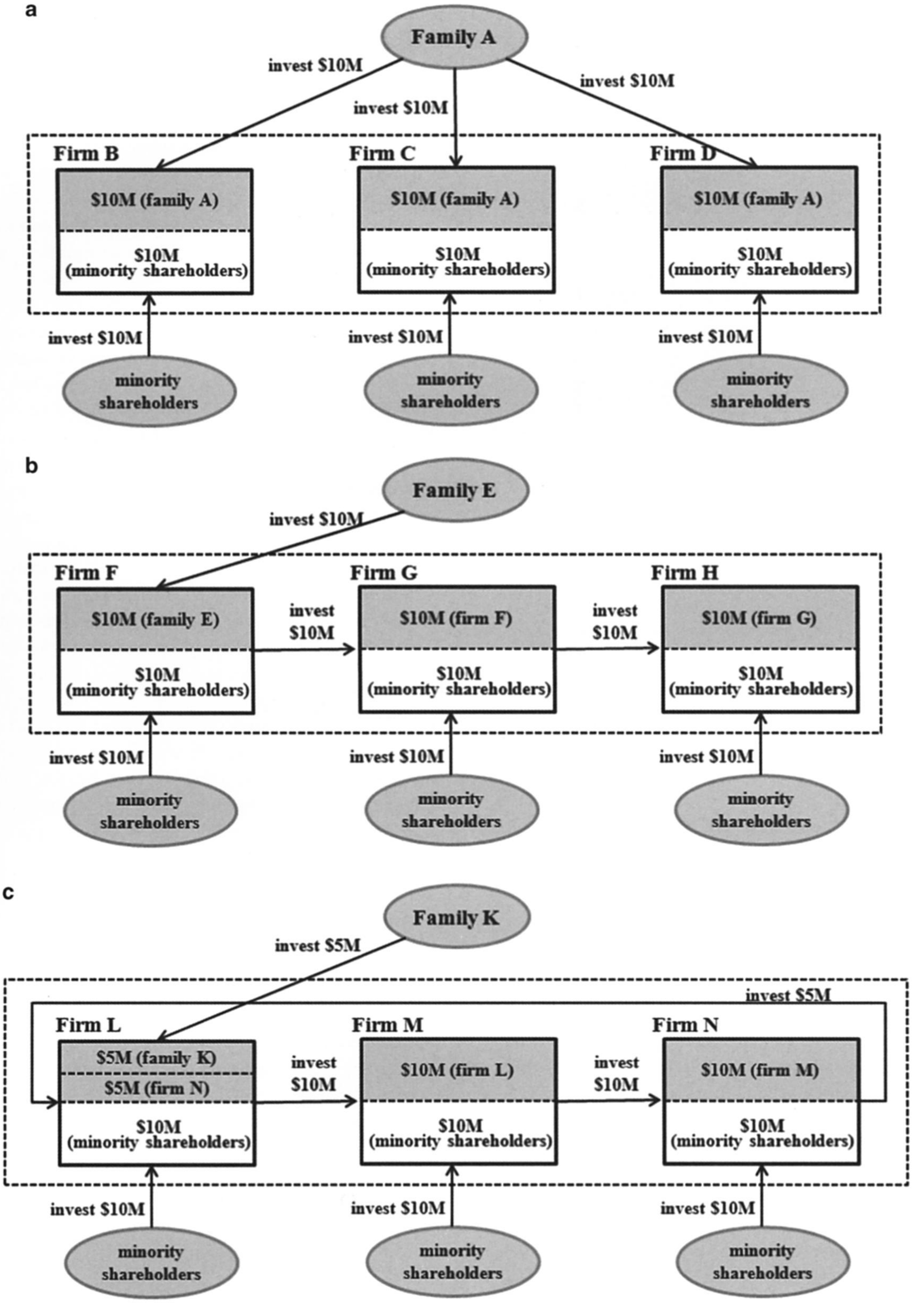

Figure 1 Examples of business group ownership structures. (a) No pyramiding and no circular shareholding; (b) Pyramiding with no circular shareholding; (c) Circular shareholding.

Table 1 An example of ownership structure

Shareholder	*Shareholding*	
	Issued by	*Ownership stake (%)*
Family 1	Firm 2	40
	Firm 3	35
Firm 2	Firm 3	25
Firm 3	Firm 5	40
Firm 4	Firm 2	20
	Firm 3	15
Firm 5	Firm 4	30

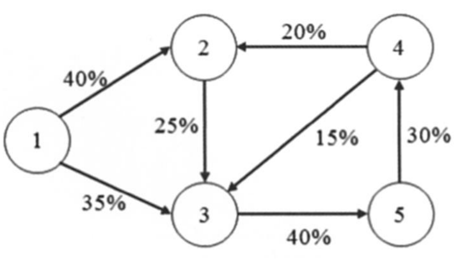

Figure 2 Network representation of ownership structure.

In an ownership network, a controlling family will be designated as node 1, which will be called the *root node* of the ownership network. Also, we make the following assumptions throughout the paper:

Assumption 1 *The root node has at least one path to reach all other nodes.*

Assumption 2 *The root node has no incoming arcs.*

The first assumption implies that a controlling family can exercise some degree of control over all member firms of its business group. The second assumption indicates that a controlling family cannot be involved in any cycles in an ownership network. In the cases we will deal with, a controlling family is a group of individuals, not an institution having shareholders; hence, Assumption 2 is satisfied by definition. In fact, most business groups in Asia, including all conglomerates in Korea, are family-owned. The properties of cash-flow and voting rights discussed later will be based on these two assumptions.

Resolving circular shareholding is equivalent to removing some arcs so that an ownership network will have no cycles. According to one well-known property of graph theory, a network has no cycle, that is, a network is acyclic if and only if it has a topological ordering where a *topological ordering* of a network with n nodes is an assignment of a distinct number $t_i \in \{1, 2, \ldots, n\}$ to every node $i \in N$ such that $t_i < t_j$ for all $(i, j) \in E$. (For the terminologies of graph theory, refer to Ahuja *et al* (1993).) With the constraint that a topological ordering should exist for an ownership network, the proposed optimization models can guarantee that all circular shareholdings will be resolved.

Another constraint we need to take into account is that the controlling family of a business group should be able to maintain its control over all member firms after circular shareholdings are cleared. Otherwise, the controlling family could foresee losing control over some firms, and thus would be reluctant to resolve circular shareholding. This constraint can be translated into the following graph–theoretical expression: There exists at least one path from the root node to all other nodes even after all cycles are removed.

The two constraints discussed above distinguish the problem of resolving circular shareholding from other existing problems. Given a network $G = (N, E)$, $G' = (N', E')$ is called a *subgraph* of G if $N' \subset N$ and $E' \subset E \cap (N' \times N')$. A combinatorial optimization problem, which addresses the removal of cycles, is the maximum acyclic subgraph problem (MASP). MASP finds a subgraph $G' = (N, E')$ such that G' contains no cycle while maximizing the cardinality of the remaining arcs. MASP is known to be NP-hard (Karp, 1972). The problem of resolving circular shareholding is similar to MASP; however, it includes another constraint, that is, all other nodes must be reachable from the root node. For example, let us consider the networks in Figure 3, which can be derived by removing some arcs from the ownership network described in Figure 2. Figures 3(a) and (b) have no cycles. Furthermore, Figure 3(b) will be optimal for MASP. However, node 4 in Figures 3(a) and (b) is not reachable from node 1, so our second constraint is not met. On the other hand, the networks in Figures 3(c) and (d) satisfy the two constraints and represent a feasible solution for the problem of resolving circular shareholding.

4. Optimization models

Now, we specify the objectives of the problem of eliminating circular shareholding. As a simple objective, we can consider minimizing the number of shareholdings cleared to resolve circular shareholding, that is, minimizing the cardinality of arcs deleted to eliminate all cycles. The minimum cardinality of deleted arcs may be used as an indicator of the degree of the complexity a business group confronts when resolving circular shareholding. Also, a set of deleted arcs with minimum cardinality can be regarded as critical arcs contributing to multiple circular shareholdings. Voting rights maximization can be another objective of optimization models. If circular shareholding was formed as an instrument for consolidating a controlling family's control over member firms, the controlling family will seek to maintain as much control as possible even after circular shareholding is resolved. Especially when the controlling family's voting rights are marginal, the family will be concerned with maximizing voting rights rather than pursuing other objectives. On the other hand, when the

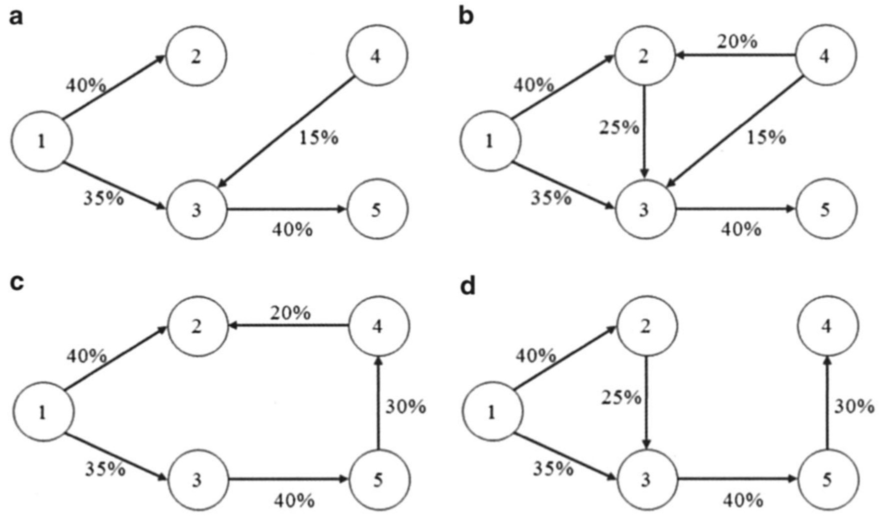

Figure 3 Ownership networks with circular shareholdings eliminated. (a) Removing arcs (2,3), (4,2) and (5,4); (b) Removing arc (5,4); (c) Removing arcs (2,3) and (4,3); (d) Removing arcs (4,2) and (4,3).

controlling family's voting rights are high enough to secure its control over member firms, we expect the family will turn to other interest, such as dividends. If circular shareholding was established from a rational decision to maximize profit, the controlling family is naturally supposed to strive for best cash-flow. Thus, the third objective of our optimization models is to maximize cash-flow rights. However, it is highly likely in the real business world that a controlling family cares about more than one objective. For example, a controlling family may desire to maximize both voting rights and cash-flow rights. A combination of voting right maximization and cash-flow right maximization might be more practical goal of the problem of resolving circular shareholding.

First, we present three optimization models, which share the two constraints proposed in the previous section but have one of the three objectives discussed above. Then, we discuss how to combine two objective functions by taking a weighted sum of them.

4.1. Minimizing the number of cleared shareholdings

Let variable w_{ij} indicate whether arc (i,j) will be removed. Arc (i,j) will be deleted if $w_{ij}=1$; otherwise, it will remain not deleted. Then, the cardinality of a set of deleted arcs can be expressed by the sum of w_{ij} across all arcs. In addition, let variable t_i denote an order for node i where $t_1=1$ and $1<t_i\leqslant n$ for $i\neq 1$. By definition of topological ordering, it must hold true that $t_i<t_j$ when arc (i,j) is not deleted for an optimal solution. However, when arc (i,j) is removed, t_i and t_j do not directly restrain each other. To ensure that there exists at least one path from the root node to all other nodes, let us define variable f_{ij} to be the number of paths going through arc (i,j). An optimization model with the objective to minimize the cardinality of a set of removed arcs while satisfying the two constraints in Section 3 can be formulated as follows:

$$
\begin{aligned}
& \min \quad \sum_{(i,j)\in E} w_{ij} \\
& \text{s.t.} \quad t_j \geqslant t_i + 1 - n w_{ij}, \quad \forall (i,j)\in E, \\
& \qquad t_1 = 1 \\
(CCSP):\ & \qquad 1<t_i\leqslant n, \forall i\in N-\{1\}, \\
& \qquad \sum_{j\in Adj(i)} f_{ij} - \sum_{k\in Adj^{-1}(i)} f_{ki} = \begin{cases} n-1, & i=1 \\ -1, & \forall i\in N-\{1\}, \end{cases} \\
& \qquad 0\leqslant f_{ij}\leqslant (n-1)\left(1-w_{ij}\right), \quad \forall (i,j)\in E, \\
& \qquad w_{ij}\in\{0,1\}, \quad \forall (i,j)\in E
\end{aligned}
$$

where $Adj(i)=\{j|(i,j)\in E\}$ and $Adj^{-1}(i)=\{j|(j,i)\in E\}$. The objective function implies that $(CCSP)$ seeks to minimize the number of removed arcs. The first through third constraints of $(CCSP)$ express that a topological ordering should exist. Next, the fourth and fifth constraints together imply that all other nodes must be reachable from the root node. The fourth constraint indicates that $(n-1)$ paths should emanate from the root node and there should be at least one path from the root node to all other nodes. The fourth constraint is the same as the constraint for the shortest path problem, which finds the shortest paths from the origin to all other nodes. Moreover, the fifth constraint expresses that any path can only pass through arcs that are not deleted.

The optimal objective function value of $(CCSP)$ will be at most the number of cycles included in an ownership network since more than one cycle may share some common arcs. For example, in Figure 2, two cycles, $(3, 5, 4, 3)$ and $(2, 3, 5, 4, 2)$, have arc $(5, 4)$ in common. Consequently, these two different cycles may be broken up by eliminating arc $(5, 4)$. Hence, the optimal objective function value of (*CCSP*) for the ownership network in Figure 2 is equal to 1. (*CCSP*) enables us to find out the minimum number of capital investment relationships that need to be cleared in order to eliminate all circular shareholdings.

Even though (*CCSP*) contains the same constraint as the shortest path problem in order to secure the existence of paths from the root node to all other nodes, an optimal solution to (*CCSP*) might not make up a tree, whereas an optimal solution for the shortest path problem without negative cycles will form a tree. The objective function of the shortest path problem includes f_{ij} for all (i, j), but the objective function of (*CCSP*) does not include any f_{ij}. Hence, an optimal solution to (*CCSP*) does not necessarily construct a tree. For example, an optimal solution to (*CCSP*) for the ownership network in Figure 2 is given in Figure 3(d). Figure 3(d) includes two paths from node 1 to node 3, $(1, 3)$ and $(1, 2, 3)$, so the network in Figure 2(d) is not a tree.

It is also worth noting that MASP can be reduced to (*CCSP*), which implies that (*CCSP*) is also NP-hard. Let us briefly describe how to transform MASP into (*CCSP*). Given a network $G = (N, E)$, an artificial node 0 is created and connected with every node $i \in N$ by adding arc $(0, i)$. Consider (*CCSP*) defined for the expanded network $\mathcal{G} = (\mathcal{N}, \varepsilon)$ where $\mathcal{N} = N \cup \{0\}$ and $\varepsilon = E \cup \{(0, i) \mid i \in N\}$. Since node 0 and all artificial arcs do not cause any new cycles, all artificial arcs will be included in an optimal solution to (*CCSP*) for $\mathcal{G}$. Hence, any solution to MASP for G can be transformed into a feasible solution to (*CCSP*) for $\mathcal{G}$. Also, an optimal solution to MASP for G can be derived from an optimal solution to (*CCSP*) for $\mathcal{G}$ by deleting node 0 and all the artificial arcs. That indicates that we can find an optimal solution to MASP for G by solving (*CCSP*) for $\mathcal{G}$.

4.2. Maximizing cash-flow rights

Before proposing another optimization model, which maximizes cash-flow rights, let us define cash-flow right in mathematical terms. The cash-flow right that a controlling family has in firm j is defined as the following:

Definition 1 Given an ownership network $G = (N, E)$, the cash-flow right in firm j, x_j, is defined as

$$x_j = \sum_{i \in Adj^{-1}(j)} s_{ij} x_i \tag{1}$$

where s_{ij} is the percentage of ownership firm (or the controlling family) i holds in firm j.

Let us call $S = (s_{ij})$ the *ownership matrix* of G. The value of s_{ij} is set to zero if firm i has no stock in firm j. We assume that $s_{ii} = 0$ for all i. Although a firm may hold stock issued by itself, that is, treasury stock, the stock gives neither cash-flow right nor voting right in the firm itself. In fact, treasury stock is regarded as unissued capital; thus, a shareholder's ownership stake is computed based on only outstanding stocks excluding treasury stock. Consequently, the assumption of $s_{ij} = 0$ does not deviate from the reality of the business world. In addition, we can naturally set $x_1 = 1$, that is, a controlling family has full cash-flow right in itself. Then, Equation (1) can be rewritten as

$$x_j = \sum_{i=2}^{n} s_{ij} x_i + c_j \tag{2}$$

where $c_j = s_{1j}$.

Definition 1 has been widely accepted as the definition of cash-flow right in many literatures (Lemmon and Lins, 2003; Claessens *et al*, 2000; La Porta *et al*, 1999). However, those literatures did not define cash-flow right in a mathematically rigorous way. As a result, there has been no proof as to whether each x_j is uniquely determined and whether it ranges from 0 to 1 in any ownership network. It is obvious that x_j is well-defined for an ownership network with no circular shareholding. However, it seems unclear that x_j is uniquely determined for an ownership network involving circular shareholding. To determine this, we need the following property for the ownership matrix:

Lemma 1 *Let* $\overline{N} = N - \{1\}$ *and* $\overline{S} = S_{\overline{N},\overline{N}}$. *Also, let* $\sigma(\overline{S}^T)$ *be the set of eigenvalues of* $\overline{S}^T$. *Let* $\rho(\overline{S}^T) = \max_{\lambda \in \sigma(\overline{S}^T)} |\lambda|$. *Then, it holds that*

(a) $(I - \overline{S}^T)$ *and* $(I + \overline{S}^T)$ *are nonsingular.*
(b) $\rho(\overline{S}^T) < 1$

where $I \in \Re^{(n-1) \times (n-1)}$ *denotes the identity matrix.*

On the basis of Lemma 1, we can show the uniqueness and validity of cash-flow right, x_j, in an arbitrary ownership network.

Theorem 1 *Each* x_j *given by Definition 1 is uniquely determined and ranges from* 0 *to* 1 *in an arbitrary ownership network.*

Let α_j represent a relative preference for the cash-flow right in firm j. Since x_1 is set to 1, the weight for the root node, α_1, has no effect on the optimal solution of the model. For simplicity, α_1 is set to zero and the sum of the weights across all firms is set to one, that is, $\sum_{j=1}^{n} \alpha_j = 1$. An optimization model that maximizes the sum of cash-flow rights while resolving all circular shareholding is as follows:

$$
\begin{aligned}
& \max \quad \sum_{j=1}^{n} \alpha_j x_j \\
& \text{s.t.} \quad x_j = \sum_{i \in Adj^{-1}(j)} \xi_{ij}, \quad \forall j \in \overline{N}, \\
& \qquad \xi_{ij} \leqslant s_{ij} x_i, \quad \forall (i,j) \in E, \\
& \qquad \xi_{ij} \leqslant s_{ij}(1 - w_{ij}), \quad \forall (i,j) \in E, \\
& \qquad x_1 = 1 \\
(FCSP): & \qquad t_j \geqslant t_i + 1 - n w_{ij}, \quad \forall (i,j) \in E, \\
& \qquad t_1 = 1, \\
& \qquad 1 \leqslant t_i \leqslant n, \quad \forall i \in \overline{N}, \\
& \qquad \sum_{j \in Adj(i)} f_{ij} - \sum_{k \in Adj^{-1}(i)} f_{ki} = \begin{cases} n-1, & i = 1 \\ -1, & \forall i \in \overline{N} \end{cases} \\
& \qquad 0 \leqslant f_{ij} \leqslant (n-1)(1 - w_{ij}), \quad \forall (i,j) \in E, \\
& \qquad w_{ij} \in \{0, 1\}, \quad \forall (i,j) \in E, \\
& \qquad x_j \geqslant 0, \quad \forall j \in \overline{N}, \\
& \qquad \xi_{ij} \geqslant 0, \quad \forall (i,j) \in E
\end{aligned}
$$

The first three constraints of (*FCSP*) express the definition of cash-flow right given in Definition 1. In particular, the second and third constraints together indicate that $\xi_{ij} \leqslant s_{ij} x_i$ if arc (i,j) is not deleted, and $\xi_{ij} = 0$ if it is deleted. Since the objective function of (*FCSP*) maximizes the weighted sum of x_j, it holds that $\xi_{ij} = s_{ij} x_i$ for an optimal solution if arc (i,j) is not deleted. The other constraints are the same as those of (*CCSP*).

Each arc's removal has a different impact on cash-flow rights. For example, let us turn to the ownership network in Figure 2. Let $\mathbf{x}^0$ denote the cash-flow right vector for the network in Figure 2. Also, let $\mathbf{x}^1$ and $\mathbf{x}^2$ denote the cash-flow right vectors for the ownership networks which are derived by removing arcs $(2, 3)$ and $(4, 2)$, respectively, from the network in Figure 2. The approximate values of $\mathbf{x}^0$, $\mathbf{x}^1$, and $\mathbf{x}^2$ are as follows:

$$\mathbf{x}^0 \simeq (1, 0.4111, 0.4611, 0.0553, 0.1844)$$

$$\mathbf{x}^1 \simeq (1, 0.4086, 0.3564, 0.0428, 0.1426)$$

$$\mathbf{x}^2 \simeq (1, 0.4000, 0.4582, 0.0550, 0.1833)$$

Then, $\|\mathbf{x}^0 - \mathbf{x}^1\|_2 \cong 0.1134$ and $\|\mathbf{x}^0 - \mathbf{x}^2\|_2 \cong 0.0115$ where $\|\cdot\|_2$ denotes the 2-norm of a vector. The difference between the ownership stakes represented by arc $(2, 3)$ and $(4, 2)$ are only 5% (=25% − 20%), but removing arc $(2, 3)$ has a much bigger impact on cash-flow rights than removing arc $(4, 2)$. Theorem 2 provides a formula for approximating the level of impact which each arc's removal causes on cash-flow rights.

Theorem 2 *Given an ownership network $G = (N, E)$, let $\mathbf{x}^G$ denote the cash-flow right vector. Let $G' = (N, E - \{(u, v)\})$ where $u \neq 1$. Then,*

$$\mathbf{x}^{G'} \leqslant \mathbf{x}^G \quad \text{and} \quad \left\| \mathbf{x}^G - \mathbf{x}^{G'} \right\|_2 \leqslant \frac{s_{uv} x_u^G}{\sigma_{\min}}$$

where $\sigma_{\min}$ is the smallest singular value of $(I - \overline{S}^T)$.

According to Theorem 2, an upper bound to the level of impact of an arc's removal is in proportion to the cash-flow right of its tail node as well as the ownership stake represented by the arc. We can expect from Theorem 2 that removal of an arc with a high ownership stake will lower cash-flow rights more than removal of an arc with a small ownership stake. In addition, removing an arc emanating from a node with high cash-flow right tends to reduce cash-flow rights more than removing an arc emanating from a node with small cash-flow right.

4.3. Maximizing voting rights

Let us introduce the definition of voting right:

Definition 2 The voting right in firm *j*, y_j, is a non-negative number such that

$$y_j = \sum_{i \in Adj^{-1}(j)} \min(s_{ij}, y_i) \tag{3}$$

where $y_1 = 1$.

Similar to cash-flow right, the voting right in the root node, that is, a controlling family itself, is defined as 1. Claessens *et al* (2000), La Porta *et al* (1999), and Faccio and Lang (2002) defined voting right in the same way as Definition 2; yet, their definition of voting right does not recognize the necessity of a non-negativity constraint for y_j. In fact, a major difference between Definition 1 and Definition 2 is that Definition 2 explicitly requires y_j to be non-negative. Without the non-negativity constraint, a solution satisfying Equation (3) may be negative. For example, let us examine an ownership network in Figure 4.

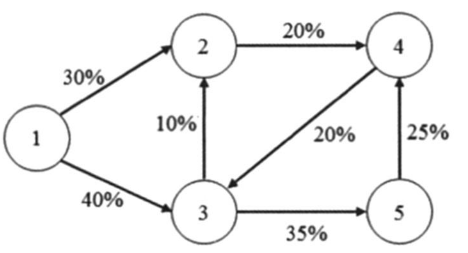

Figure 4 Example ownership network.

By Definition 2, the following Equations together define the voting right vector, $\mathbf{y}$, for the ownership network:

$$\begin{aligned} y_2 &= 0.3 + \min(0.1, y_3) \\ y_3 &= 0.4 + \min(0.2, y_4) \\ y_4 &= \min(0.2, y_2) + \min(0.25, y_5) \\ y_5 &= \min(0.35, y_3) \end{aligned} \tag{4}$$

If we ignore the non-negativity constraint of Definition 2, there are two solutions to the system of Equation (4). One of the solutions is

$$\mathbf{y} = (1, -0.4, -0.7, -1.1, -0.7)^T$$

which includes negative components. The other solution to the system of Equation (4) is

$$\mathbf{y} = (1, 0.4, 0.6, 0.45, 0.35)^T$$

which satisfies Definition 2. From Theorem 1, we know that cash-flow right cannot be negative even though its definition does not explicitly include a non-negativity constraint. However, this is not the case for voting right. The definition of voting right needs a non-negativity constraint in order to exclude any negative solutions to the system of Equation (3).

Despite the widespread evaluation of voting rights in many researches, the calculation of voting rights has mostly been applied to simple cases not involving complex ownership structures such as circular shareholdings. Hence, no rigorous investigation has been carried out to determine whether voting rights are well-defined in an arbitrary ownership network. The following theorem confirms the uniqueness and validity of each voting right, y_j, defined by Definition 2.

Theorem 3 *Given an arbitrary ownership network G, each voting right, y_i, from Definition 2 is uniquely determined and ranges from 0 to 1.*

Lemma 2 in the Appendix and Theorem 3 provide a convenient way for calculating voting rights: First, we construct the system of Equation (3). Next, we identify an cycle $\mathcal{C} = (j_1, \ldots, j_{r+1})$ and find an arc $(u, v) \in \mathcal{C}$ such that $s_{uv} = \min_{1 \leqslant k \leqslant r} s_{j_k j_{k+1}}$. Third, we replace $\min(y_u, s_{uv})$ in the system of Equation (3) with s_{uv}. After repeating the procedure until there is no recursive equation in the system, we can calculate the voting rights for all nodes in an ownership network.

Now, let us check whether the gap between voting right and cash-flow right can be mathematically demonstrated. Comparing the definition of cash-flow right and voting right, each term in Definition 1 is the product of a cash-flow right and a percentage of ownership. On the other hand, each term in Definition 2 is the smaller of a voting right and a percentage of ownership. Since cash-flow rights, voting rights, and percentages of ownership are usually less than 1, we can conjecture that voting rights, which are based on the minimum of two values less than 1, will not be less than cash-flow rights, which are based on the product of two values less than 1. The following theorem confirms the conjecture:

Theorem 4 *Given an ownership network G with an ownership matrix S, let $\mathbf{x}^G$ and $\mathbf{y}^G$ denote the cash-flow right vector and voting right vector, respectively. Then,*

$$\mathbf{y}^G \geqslant \mathbf{x}^G$$

The last optimization model, which maximizes a weighted sum of voting-rights while resolving circular shareholding, can be constructed in a way similar to (*FCSP*). Let α_j represent a relative preference for the voting right in firm *i*. Since $y_1 = 1$, α_1 can be set to 0 and the sum of weights for all firms, $\sum_{j=1}^{n} \alpha_j$, is set to one. A mathematical programming model for maximizing a weighted sum of voting rights is formulated as follows:

$$\begin{aligned} &\max && \sum_{j=1}^{n} \alpha_j y_j \\ &\text{s.t.} && y_j = \sum_{i \in Adj^{-1}(j)} \eta_{ij}, \quad \forall j \in \overline{N}, \\ &&& \eta_{ij} \leqslant y_i, \quad \forall (i,j) \in E, \\ &&& \eta_{ij} \leqslant s_{ij}(1 - w_{ij}), \quad \forall (i,j) \in E, \\ &&& y_1 = 1 \\ &(VCSP): && t_j \geqslant t_i + 1 - n w_{ij}, \forall (i,j) \in E, \\ &&& t_1 = 1, \\ &&& 1 \leqslant t_i \leqslant n, \quad \forall i \in \overline{N}, \\ &&& \sum_{j \in Adj(i)} f_{ij} - \sum_{k \in Adj^{-1}(i)} f_{ki} = \begin{cases} n-1, & i = 1 \\ -1, & \forall i \in \overline{N} \end{cases} \\ &&& 0 \leqslant f_{ij} \leqslant (n-1)(1 - w_{ij}), \quad \forall (i,j) \in E, \\ &&& w_{ij} \in \{0, 1\}, \quad \forall (i,j) \in E, \\ &&& y_j \geqslant 0, \quad \forall j \in \overline{N}, \\ &&& \eta_{ij} \geqslant 0, \quad \forall (i,j) \in E \end{aligned}$$

The first four constraints of (*VCSP*) ensure that each y_j represents a voting right. The other constraints are the same as those of (*FCSP*).

4.4. Combined model

Each of the three models presented so far has one goal of minimizing the cardinality of removed arcs, maximizing cash-flow rights, or maximizing voting rights. However, a controlling family might be concerned with more than one goal. For example, it is very likely that it will seek to maximize both voting rights and cash-flow rights when resolving circular shareholding. Also, a controlling family may pursue maximization of cash-flow rights while eliminating as few arcs in an ownership network as possible. One way to handle more than one goal is to use multi-objective programming, which optimizes a weighted sum of multiple objective functions. Several multi-objective programming models can be derived by combining two objective functions out of the three proposed models. In particular, a combination of (*FCSP*) and (*VCSP*) (referred to as (*FCSP* + *VCSP*)) will be helpful for identifying arcs whose removal leads to the least reduction in the weighted sum of cash-flow rights and voting rights. The objective function of (*FCSP* + *VCSP*) can be written as the following:

$$\max \quad \omega_1 \left(\sum_{j=1}^{n} \alpha_j x_j \right) + \omega_2 \left(\sum_{j=1}^{n} \alpha_j y_j \right)$$

where ω_1 and ω_2 represent the relative importance for cash-flow rights and voting rights, respectively. The constraints of (*FCSP* + *VCSP*) are obtained by merging those of (*FCSP*) and (*VCSP*).

5. Applications to real-world data

To apply the proposed models to Korean business groups, we focused on the 11 largest business groups that, as of 2007, were subject to a ceiling on total investment in member firms. These 11 business groups accounted for around 50% of the total stock market capitalization in Korea as of 2007 (Korea Exchange, 2008). Since four out of these business groups had no circular shareholding, we collected data for the other seven business groups. All data was obtained from the online financial holdings information disclosure system supplied by the Korean Fair Trade Commission (Korea Fair Trade Commission, 2011), the government unit responsible for business group regulation. The extracted data details the ownership structure of every firm belonging to a business group as of 2007, and includes the percentages of common stock held by a controlling family and other member firms. (All of the Korean firms follow a one-share-one-vote structure.) When counting the total shares held by a controlling family, we considered the whole family as one unit rather than distinguished among individual family members. Furthermore, α_j, a preference for each firm in the objective functions of (*FCSP*), (*VCSP*), and (*FCSP* + *VCSP*), is set to each firm's relative size of shareholder's equity compared to the total sum of shareholders' equity aggregated across all member firms.

Table 2 includes basic information for each of the seven major conglomerates in Korea. The second and third columns represent the number of nodes and arcs in the ownership network for each conglomerate. The fourth column, average degree, represents the ratio of the number of arcs to the number of nodes, that is, the average number of other member firms one firm invests in. The next columns show the sum of cash-flow rights, the sum of voting rights, and the ratio of the sum of voting rights to the sum of cash-flow rights. The sum of cash-flow rights is computed in two different ways. The unweighted sum is calculated by simply adding together the cash-flow rights of all member firms. On the other hand, the weighted sum is obtained by multiplying each firm's cash-flow right by the firm's weight and aggregating it across all member firms. Note that the sum of cash-flow rights does not include the cash-flow right in a controlling family itself. The weighted and unweighted sum of voting rights are computed in the same way as those of cash-flow rights are. When comparing one conglomerate with another with respect to cash-flow rights or voting rights, the weighted sum is more useful because the unweighted sum tends to be bigger for large conglomerates and does not reflect the difference in the number of member firms.

Among the seven conglomerates in Table 2, Lotte Group and Samsung Group have the most complicated ownership networks in terms of the number of nodes and arcs. Also, these two conglomerates are the most densely connected among member firms in terms of average degree. However, we can see that the average degrees, as a whole, are small relative to the number of nodes in the ownership networks. This implies that the ownership networks are overall sparse, and therefore circular shareholding may be resolved by eliminating a small number of arcs. When comparing cash-flow rights with voting rights, a significant gap between cash-flow rights and voting rights is common to all the conglomerates. In particular, in terms of the ratio of voting rights to cash-flow rights, Samsung Group and Hyundai Motor Group exhibit the biggest discrepancies between their respective voting rights and cash-flow rights while Lotte Group has the smallest discrepancy. In fact, in

Table 2 Large Korean conglomerates

Conglomerate	*Nodes*	*Arcs*	*Average degree*	*Unweighted sum*			*Weighted Sum*		
				CF rights	*V rights*	*V/CF*	*CF rights*	*V rights*	*V/CF*
Hanjin	11	28	2.55	2.00104	3.39700	1.70	0.15755	0.24727	1.57
Hanwha	10	19	1.90	1.76009	3.98240	2.26	0.21335	0.39218	1.84
Hyundai Motor	5	10	2.00	0.35476	1.20060	3.38	0.08117	0.28214	3.48
Hyundai Heavy Ind.	8	10	1.25	0.72658	1.92900	2.65	0.12249	0.25491	2.08
Lotte	22	99	4.50	12.51781	15.84050	1.27	0.64484	0.73772	1.14
Samsung	19	76	4.00	1.24142	4.72790	3.81	0.06100	0.20023	3.28
SK	5	8	1.60	0.67666	1.18330	1.75	0.05738	0.15664	2.73

Table 3 Results from the optimization models

Conglomerates	*Cardinality minimization (CCSP)*						
		Cash-flow right			*Voting right*		
	DArcs	*After*	*Change*	*Change%*	*After*	*Change*	*Change%*
Hanjin	2	0.14986	0.00770	4.88	0.20080	0.04647	18.79
Hanwha	1	0.21202	0.00133	0.62	0.36972	0.02247	5.73
Hyundai Motor	1	0.07430	0.00687	8.46	0.19403	0.08811	31.23
Hyundai Heavy Ind.	1	0.11658	0.00591	4.82	0.13371	0.12120	47.55
Lotte	8	0.60459	0.04025	6.24	0.68352	0.05421	7.35
Samsung	8	0.04347	0.01753	28.73	0.08309	0.11714	58.50
SK	1	0.02711	0.03028	52.76	0.03829	0.11835	75.56
Conglomerates	*Cash-flow right maximization (FCSP)*						
		Cash-flow right			*Voting right*		
	DArcs	*After*	*Change*	*Change%*	*After*	*Change*	*Change%*
Hanjin	7	0.15509	0.00246	1.56	0.23753	0.00977	3.94
Hanwha	1	0.21202	0.00133	0.62	0.36972	0.02247	5.73
Hyundai Motor	2	0.07728	0.00389	4.79	0.22381	0.05833	20.67
Hyundai Heavy Ind.	1	0.11658	0.00591	4.82	0.13371	0.12120	47.55
Lotte	11	0.63376	0.01107	1.72	0.72939	0.00833	1.13
Samsung	17	0.05983	0.00117	1.92	0.18094	0.01929	9.63
SK	2	0.05652	0.00087	1.51	0.14622	0.01043	6.66
Business group	*Multi-objective optimization (FCSP + VCSP)*						
		Cash-flow right			*Voting right*		
	DArcs	*After*	*Change*	*Change%*	*After*	*Change*	*Change%*
Hanjin	7	0.15509	0.00246	1.56	0.23753	0.00974	3.94
Hanwha	1	0.21202	0.00133	0.62	0.36972	0.02247	5.73
Hyundai Motor	2	0.07728	0.00389	4.79	0.22381	0.05833	20.67
Hyundai Heavy Ind.	1	0.11658	0.00591	4.82	0.13371	0.12120	47.55
Lotte	11	0.63376	0.01107	1.72	0.72939	0.00833	1.13
Samsung	13	0.05981	0.00120	1.96	0.18295	0.01728	8.63
SK	2	0.05652	0.00087	1.51	0.14622	0.01043	6.66
Business group	*Voting right maximization (VCSP)*						
		Cash-flow right			*Voting right*		
	DArcs	*After*	*Change*	*Change%*	*After*	*Change*	*Change%*
Hanjin	3	0.14958	0.00797	5.06	0.23854	0.00873	3.53
Hanwha	4	0.20940	0.00395	1.85	0.37292	0.01926	4.91
Hyundai Motor	2	0.07728	0.00389	4.79	0.22381	0.05833	20.67
Hyundai Heavy Ind.	1	0.11658	0.00591	4.82	0.13371	0.12120	47.55
Lotte	11	0.63335	0.01149	1.78	0.73009	0.00763	1.03
Samsung	13	0.05981	0.00120	1.96	0.18295	0.01728	8.63
SK	2	0.05652	0.00087	1.51	0.14622	0.01043	6.66

terms of weighted sum, Samsung Group and Hyundai Motor Group maintain relatively small cash-flow rights while Lotte Group retains the largest cash-flow rights.

Table 3 shows the results obtained by applying the proposed optimization models. When applying the combined model (*FCSP* + *VCSP*), we set the weight of cash-flow rights and voting rights to 0.7 and 0.3, respectively. (Since the sum of voting rights tends to be larger than the sum of cash-flow rights, a higher weight of cash-flow rights than voting rights will be helpful to make a balance between cash-flow maximization and voting right maximization.) Table 3 shows the number of removed arcs and changes in cash-flow rights and voting rights after each model is applied. The *DArcs* column indicates the number of arcs removed by each model, while the *After* column represents the weighted sum of cash-flow rights or voting rights after circular shareholding is resolved using each model. Lastly, *Change* and *Change%* columns represent the absolute and relative amount of decrease in the sum of cash-flow rights or voting rights, respectively.

The first finding from the application results is that the minimum number of arcs to be removed for resolving circular shareholding, which is found by (*CCSP*), is small. The number of arcs eliminated by (*CCSP*) ranges from one to eight. Of the seven conglomerates, five can resolve circular shareholding by removing one or two shareholdings among member firms. The other two conglomerates, Lotte and Samsung, need to eliminate eight arcs, a relatively small number compared with the total number of arcs in their ownership networks. However, the other three models may remove significantly more arcs than (*CCSP*). For example, in the case of Hanjin Group, (*FCSP*) and (*FCSP* + *VCSP*) remove seven arcs while (*CCSP*) removes only two arcs. Also, in the case of Samsung Group, (*FCSP*) removes 17 arcs, which is more than twice the number of arcs removed by (*CCSP*). (One exception is Hyundai Heavy Industries Group, in which all the four optimization models share one unique solution. In fact, Hyundai Heavy Industries Group has only one way to resolve circular shareholding while satisfying the constraints.) These results indicate that circular shareholdings in the conglomerates tend to share a small number of common arcs, which are critical for maintaining a controlling family's cash-flow and voting rights. While (*CCSP*) eliminates those common arcs to minimize the cardinality of removed arcs, the other three models tend to eliminate other arcs not shared among multiple circular shareholdings.

The second finding is that cardinality minimization in resolving circular shareholding may lead to a significant decrease in cash-flow and voting rights. Figure 5 shows how cash-flow and voting rights vary across the four models. Overall, (*CCSP*) considerably lowers cash-flow and voting rights. (One exception is Hanwha Group for which (*CCSP*) and (*FCSP*) happened to have the same optimal solution.) The reason is that (*CCSP*) considers neither the size of an ownership stake represented by a removed arc nor the impact of eliminated ownership on cash-flow rights and voting rights. Thus, (*CCSP*) may remove some core arcs that are crucial for maintaining a controlling family's cash-flow and voting rights. For instance, in the case of SK Group, (*CCSP*) decreases cash-flow rights by 52.8% and voting rights by 75.6%. In contrast, the other three models lower cash-flow rights by a mere 1.5% and voting rights by 6.7%. Hence, the other three models should be applied when cash-flow rights or voting rights are of primary concern rather than the number of cleared shareholdings.

The third finding is that there are some variations among business groups in the decrease of cash-flow rights and voting rights. Looking at cash-flow rights, (*FCSP*) reduces Hanwha Group's cash-flow rights by only 0.6% while it reduces Hyundai Motor and Hyundai Heavy Industries Groups' cash-flow rights by 4.8%. The variations in the decrease of cash-flow rights can be attributed to two factors: the size and the importance of eliminated shareholding. When the size of an ownership stake is small and the stock ownership is associated with a member firm that has a small weight in the objective functions, the elimination of the ownership will have a marginal impact on cash-flow rights. In the case of Hanwha Group, the ownership stake represented by the removed arc is insignificant, which leads to a small decrease in cash-flow rights. On the other hand, in the case of Hyundai Motor Group, the sum of ownership stakes across the removed arcs is larger than any other business groups', which accounts for Hyundai Motor Group having the biggest decrease in cash-flow rights. In the case of Hyundai Heavy Industries Group, the ownership stake represented by the removed arc is not big, but the ownership is associated with a major firm of largest weight. To resolve circular shareholding while satisfying the constraints, Hyundai Heavy Industries Group has no other alternative but to clear a shareholding in the major member firm, which leads to a relatively large decrease in cash-flow rights.

The variations among business groups in the decrease in voting rights is equally revealing. The decrease in the voting rights of Hanjin Group and Lotte Group were relatively small while the voting rights of Hyundai Motor Group and Hyundai Heavy Industries Group were severely reduced. To understand what causes the variation among the conglomerates in the magnitude of the decrease in voting rights, let us compare Hyundai Motor Group and SK Group that have almost the same number of nodes and arcs in their ownership networks but differ considerably in the decrease in voting rights. SK Group's circular shareholding involves a member firm, SK C&C, in which its controlling family holds a 55% ownership stake. Also, SK C&C has a very low weight in the objective function due to its small size of shareholder's equity, which implies that the decrease in the voting right in SK C&C does not significantly impact the total sum of voting rights. The optimal solution to (*VCSP*) eliminates all incoming arcs to SK C&C, lowering the voting right in the member firm. However, the voting right in SK C&C still remains sufficiently high due to a big ownership stake held by the controlling family. Thus, the voting rights in other member firms, which are owned by SK C&C, are not affected. (Note that, in Definition 2, $\min(s_{ij}, y_i) = \min(s_{ij}, y'_i)$ if $y_i > y'_i \geqslant s_{ij}$.) Therefore, the reduction in the total sum of voting rights is not big since the voting rights are lowered only in SK C&C, which has a small weight in the objective function.

In contrast, Hyundai Motor Group differs from SK Group in two ways. First, Hyundai Motor Group's controlling family does not have a high percentage of ownership in any member firms. Second, except one major firm, other member firms have a similar size of shareholder's equity, so the weights of member firms in the objective function are much more equally distributed in Hyundai Motor Group than in SK Group. Because of these features of Hyundai Motor Group, the removal of an arc

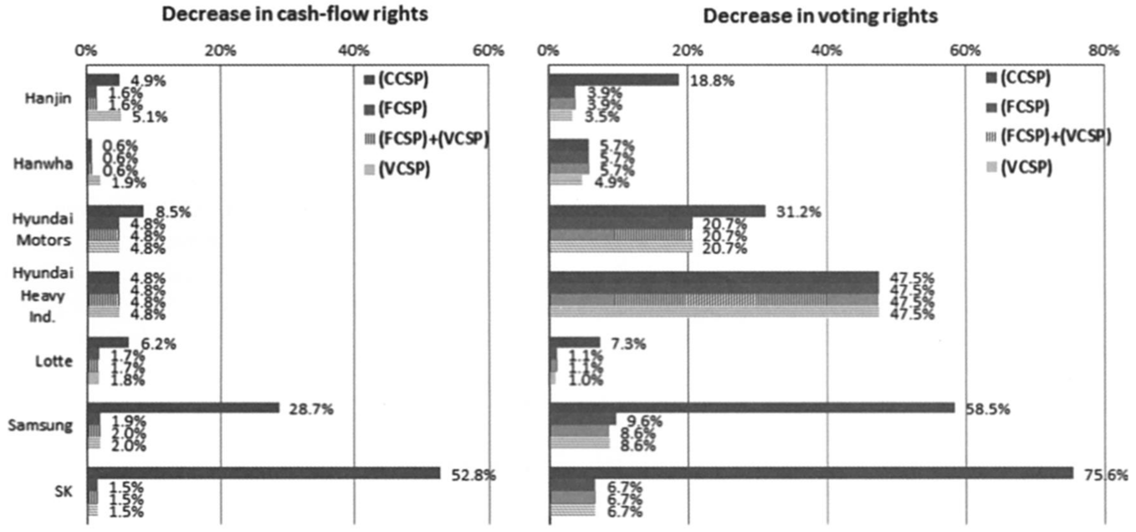

Figure 5 Decrease in cash-flow and voting rights after circular shareholding is eliminated.

(i, j) leads to a decrease in the voting right in member firm j, which in turn reduces the voting rights in a major firm and other member firms owned by firm j because the voting right in member firm j does not remain high enough.

The fourth finding is that resolving circular shareholding tends to have a bigger impact on voting rights than cash-flow rights in terms of both absolute and relative magnitude. As seen in Figure 5, the sum of voting rights decreased much more than the sum of cash-flow rights in most cases. Besides, even (*VCSP*), which seeks to maximize voting rights, produces optimal solutions that reduce voting rights more than cash-flow rights in terms of absolute magnitude. This phenomenon may indicate that circular shareholding inflates voting rights more than cash-flow rights. Understanding what causes this phenomenon will need further mathematical analysis on the properties of cash-flow and voting rights.

Interestingly, in the case of Lotte Group, cash-flow rights were reduced more than voting rights. Lotte Group has the highest cash-flow and voting rights among the seven conglomerates. Also, it has the smallest discrepancy between voting rights and cash-flow rights in terms of a ratio of voting rights to cash-flow rights. For a majority of member firms in the Lotte Group, the controlling family's voting rights reach around 70%. Such high voting rights in the Lotte Group member firms can be explained by two features in its ownership network. One feature is that the ownership in member firms of the Lotte Group is heavily concentrated in its controlling family and two member firms, one of which is 100% owned by the controlling family and consequently is not involved in any circular shareholding. In fact, Lotte Group's controlling family has direct ownership in 17 member firms. For 9 of the 17-member firms, the ownership stake held by the controlling family is more than 20%, which is high compared with the controlling families of other conglomerates. The other feature is that major member firms in the Lotte Group are densely connected by other member firms as its highest average degree in Table 2 indicates. Because of these features, removing an arc (i, j) slightly lowers the voting right in firm j but the voting right in firm j still remains higher than the ownership stake represented by any arc outgoing from firm j. In this case, by Definition 2, the impact on voting rights is not propagated beyond firm j. Consequently, the sum of voting rights was marginally reduced after circular shareholding was resolved. However, by definition, removing an arc (i, j) lowers not only the cash-flow right in firm j but also the cash-flow rights in all other firms reachable from firm j. Hence, circular shareholding may have a bigger impact on cash-flow rights more than voting rights especially when the controlling family's voting rights in member firms remains sufficiently high even after circular shareholding is eliminated.

Our final finding is that (*FCSP* + *VCSP*) can be an effective choice especially when maximization of cash-flow rights conflicts with maximization of voting rights. For most of the conglomerates, a choice between (*FCSP*) and (*VCSP*) appears to have a marginal impact on cash-flow and voting rights compared with (*CCSP*). As seen in Figure 5, the decrease in voting rights resulting from (*VCSP*) is equal to or slightly lower than that resulting from (*FCSP*). Also, for most of the conglomerates, the decrease in cash-flow rights resulting from (*FCSP*) is equal to or slightly smaller than that resulting from (*VCSP*). It becomes clearer when the two models are compared in the case of the Lotte Group, which has many alternatives in removing arcs due to its complicated ownership network. In the Lotte Group, (*FCSP*) and (*VCSP*) showed only a small difference in terms of cash-flow rights and voting rights. However, (*FCSP* + *VCSP*) can be a good choice for balancing cash-flow rights and voting rights especially when (*FCSP*) compromises voting rights or (*VCSP*) compromises cash-flow rights relatively much. In the case of the Hanjin Group, (*VCSP*) reduced cash-flow rights significantly more than (*FCSP*). The combined model, (*FCSP* + *VCSP*), showed the same amount of decrease in cash-flow rights as (*FCSP*), while reducing voting rights slightly more than (*VCSP*). Also, in the case of the Samsung Group, (*FCSP* + *VCSP*) achieved the same level of voting rights as (*VCSP*), while lowering cash-flow rights slightly more than (*FCSP*).

6. Conclusion

In this paper, we proposed optimization approaches for resolving circular shareholding among member firms of a family-owned business group. Of the four proposed models, the first one eliminates all circular shareholdings by removing a set of arcs with minimum cardinality. The second model resolves circular shareholding whereby the sum of cash-flow rights a controlling family has in its member firms is maximized. The third model identifies a set of arcs that eliminates all circular shareholdings while maximizing the sum of voting rights. The last model combines two objectives of maximizing cash-flow rights and maximizing voting rights.

The results from the application of the models to Korean conglomerates showed that circular shareholding can be resolved by removing a relatively small number of arcs. Also, the elimination of circular shareholding has a relatively small impact on cash-flow rights. In most cases, however, voting rights are significantly affected by resolving circular shareholding.

Finally, this research will provide valuable information and tools not just for business groups who have a plan to eliminate circular shareholding, but also for decision makers who establish governmental policies regarding conglomerate's ownership structure. The proposed models can be a basis from which other sophisticated optimization models, such as models for maximizing cash-flow rights or voting rights by adding new shareholdings, can be developed. Such models can help aid progress in the difficult task of addressing circular shareholding practices, especially in the case of Korean family business groups.

Acknowledgements—We would like to thank the two anonymous referees and Professor McDonald for their valuable comments on the earlier version of this manuscript. This work was partially supported by the Ajou University research fund. H Shin would like to gratefully acknowledge the grant support from National Research Foundation of the Korean Government (2013R1A1A3010440/2012-0000994).

References

Ahuja RK, Magnanti TL and Orlin JB (1993). *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall: Englewood Cliffs, NJ.

Almeida H, Park S, Subramaniam M and Wolfenzon D (2008). *Beyond cash flow and voting rights: Valuation and performance of firms in complex ownership structures*. Working paper, University of Illinois at Urbana-Champaign and Stern School of Business at NYU.

Baek J, Kang J and Park K (2004). Corporate governance and firm value: Evidence from the Korean financial crisis. *Journal of Financial Economics* **71**(2): 265–313.

Bauer R, Guenster N and Otten R (2004). Empirical evidence on corporate governance in Europe: The effect on stock returns, firm value and performance. *Journal of Asset Management* **5**(2): 91–104.

Claessens S, Djankov S and Lang LHP (2000). The separation of ownership and control in East Asian corporations. *Journal of Financial Economics* **58**(1–2): 81–112.

Faccio M and Lang L (2002). The ultimate ownership of Western European corporations. *Jounral of Financial Economics* **65**(3): 365–395.

Golub GH and Van Loan CF (1996). *Matrix Computations*, 3rd edn. The Johns Hopkins University Press: Baltimore, MD.

Gompers P, Ishii J and Metrick A (2003). Corporate governance and equity prices. *The Quarterly Journal of Economics* **118**(1): 107–155.

Haggard S, Lim W and Kim E (2003). *Economic Crisis and Corporate Restructuring in Korea*. Cambridge University Press: Cambridge, UK.

Joh S (2003). Corporate governance and firm profitability: Evidence from Korea before the economic crisis. *Journal of Financial Economics* **68**(2): 287–322.

Karp RM (1972). Reducibility among combinatorial problems. In: Miller RE and Thatcher JW (eds). *Complexity of Computer Computations*. Plenum: New York, pp 85–103.

Korea Exchange (2008). Press release from Korea Exchange, http://www.krx.co.kr, accessed 15 January 2014.

Korea Fair Trade Commission (2010). *Statistical Yearbook*. Korea Fair Trade Commission: Seoul.

Korea Fair Trade Commission (2011). Online Provision of Enterprises Information System. http://groupopni.ftc.go.kr, accessed 20 January 2011.

La Porta R, Lopez-de-Silanes F and Shleifer A (1999). Corporate ownership around the world. *The Journal of Finance* **54**(2): 471–518.

Lemmon ML and Lins KV (2003). Ownership structure, corporate governance, and firm value: Evidence from the East Asian financial crisis. *The Journal of Finance* **58**(4): 1445–1468.

Lim Y (2006). *Understanding of circular shareholdings and govermental policy direction*. Korea Development Institute Policy Forum 169, Korea Development Institute.

Meyer CD (2000). *Matrix Analysis and Applied Linear Algebra*. SIAM: Philadelphia, PA.

Park S (2006). Prohibition of circular shareholding as an alternative to the ceiling on total amount shareholding in other domestic companies. *Journal of Korean Competition Law* **14**(1): 17–57.

Appendix

Proof of Lemma 1 First, let us show that $(I-\bar{S}^T)$ is nonsingular. For contradiction, suppose that $(I-\bar{S}^T)$ is singular. Then, there exists $\mathbf{z}=(z_2,\ldots,z_n)^T\neq 0$ such that

$$\left(I-\bar{S}^T\right)\mathbf{z}=0 \tag{A.1}$$

Let $k^*=\operatorname{argmax}\{|z_i|\,|i=2,\ldots,n\}$. The k^*-th component of Equation (A.1) is

$$z_{k^*}-\sum_{i=2}^{n}s_{i,k^*}z_i=0 \tag{A.2}$$

By Assumption 1, $G=(N,E)$ has at least one path from the root node to k^*. Let $(1,k_1,\ldots,k_{l-1},k_l=k^*)$ denote one of the paths where l is the number of arcs on the path. After dividing both sides by z_{k^*}, Equation (A.2) is rewritten as

$$1 = \sum_{i=2}^{n} s_{i,k^*} \frac{z_i}{z_{k^*}} \tag{A.3}$$

By definition, $\overline{S}^T$ is a non-negative matrix and $\sum_{i=2}^{n} s_{ij}$ is less than or equal to 1 for all j. If $l=1$, then $c_{k^*}>0$ and the right-hand side of Equation (A.3) is less than 1 because $\sum_{i=2}^{n} s_{i,k^*} \leqslant 1-c_{k^*}<1$ and $|z_i/z_{k^*}| \leqslant 1$. This is contradictory, which implies that $(I-\overline{S}^T)$ is non-singular. On the other hand, suppose that $l>1$. Then, to satisfy (A.3), it must hold that $z_i = z_{k^*}$ for all $i \in \overline{N}$ such that $s_{i,k^*}>0$. Consequently, $z_{k_{l-1}} = z_{k^*}$. By applying the same argument repeatedly, we can obtain $z_{k_1} = z_{k^*}$. Like Equation (A.3), the following equation can be derived for k_1:

$$1 = \sum_{i=2}^{n} s_{i,k_1} \frac{z_i}{z_{k_1}} \tag{A.4}$$

Since $c_{k_1}>0$, it holds that $\sum_{i=2}^{n} s_{i,k_1}<1$, which implies that the right-hand side of Equation (A.4) is less than 1, making a contradiction. Therefore, $(I-\overline{S}^T)$ is non-singular.

Next, by applying a technique similar to the above, let us show that $(I+\overline{S}^T)$ is non-singular. Suppose that $(I+\overline{S}^T)$ is singular. Then, there exists $\mathbf{z}=(z_2, \ldots, z_n)^T \neq 0$ such that

$$\left(I+\overline{S}^T\right)\mathbf{z} = 0 \tag{A.5}$$

Let $k^* = \text{argmax}\,\{|z_i|\,|i=2,\ldots,n\}$. The k^*-th component of Equation (A.5) is

$$z_{k^*} + \sum_{i=2}^{n} s_{i,k^*} z_i = 0 \tag{A.6}$$

After dividing both sides by z_{k^*}, Equation (A.6) is rewritten as

$$-1 = \sum_{i=2}^{n} s_{i,k^*} \frac{z_i}{z_k^*} \tag{A.7}$$

Let $(1, k_1, \ldots, k_{l-1}, k_l = k^*)$ denote one of the paths in $G=(N,E)$ from the root node to node k^* where l is the number of arcs on the path. If $l=1$, the right-hand side of Equation (A.7) is greater than -1 since $\sum_{i=2}^{n} s_{i,k^*} \leqslant 1-c_{k^*}<1$ and $|z_i/z_{k^*}| \leqslant 1$ for all $i \in \overline{N}$, which is contradictory. On the other hand, if $l>1$, it must be true that $|z_i| = |z_{k^*}|$ for all $i \in \overline{N}$ such that $s_{i,k^*}>0$. Hence, $|z_{k_{l-1}}| = |z_{k^*}|$. By applying the same argument repeatedly, we can obtain $|z_{k_1}| = \cdots = |z_{k_{l-1}}| = |k|$. Like Equation (A.7), the following equation can be derived for the k_1-th component of Equation (A.5):

$$-1 = \sum_{i=2}^{n} s_{i,k_1} \frac{z_i}{z_{k_1}} \geqslant -\sum_{i=2}^{n} s_{i,k_1} \tag{A.8}$$

Since $c_{k_1}>0$, it holds that $\sum_{i=2}^{n} s_{i,k_1}<1$, which implies that Equation (A.8) is contradictory. Therefore, $(I+\overline{S}^T)$ is non-singular.

Finally, let us show that $\rho(\overline{S}^T)<1$. By Gerschgorin's theorem (Meyer, 2000), we obtain

$$|\lambda| \leqslant \max_j \left(\sum_{i \in \overline{N}-\{j\}} s_{ij} \right) \leqslant 1 \quad \text{for all } \lambda \in \sigma\left(\overline{S}^T\right) \tag{A.9}$$

Let $\lambda_{\max} = \max\{\lambda \mid \lambda \in \sigma(\overline{S}^T) \cap \Re\}$. From the properties of non-negative matrices (Meyer, 2000), we get $\rho(\overline{S}^T) = \lambda_{\max}$. Since $(I-\overline{S}^T)$ is non-singular, we also see that $1 \notin \sigma(\overline{S}^T)$, which, together with inequality (A.9), indicates that $\lambda_{\max} = \rho(\overline{S}^T)<1$. □

Proof of Theorem 1 First, it is obvious from Lemma 1 that the non-singularity of $(I-\overline{S}^T)$ leads to determining x_j uniquely. Next, we need to show $0 \leqslant x_j \leqslant 1$ for every $j \in \overline{N}$. Since $\rho(\overline{S}^T)<1$ from Lemma 1, Equation (2) can be rewritten as

$$\mathbf{x} = \left(I-\overline{S}^T\right)^{-1}\mathbf{c} = \left(I+\overline{S}^T+\left(\overline{S}^T\right)^2+\cdots\right)\mathbf{c} \tag{A.10}$$

Since $\overline{S}^T$ is a non-negative matrix, we see from the above Equation (A.10) that $x \geqslant 0$. Finally, to show that $x_j \leqslant 1$, let $x_{k^*} = \max\{x_i | i=2,\ldots,n\}$. Suppose that $x_{k^*}>1$. Then, Equation (2) for x_{k^*} is written as

$$1 = \sum_{i=2}^{n} \left(s_{i,k^*}\right)\left(\frac{x_i}{x_{k^*}}\right) + \left(c_{k^*}\right)\left(\frac{1}{x_{k^*}}\right) \tag{A.11}$$

If $c_{k^*}>0$, we get

$$\sum_{i=2}^{n} \left(s_{i,k^*}\right)\left(\frac{x_i}{x_{k^*}}\right) + \left(c_{k^*}\right)\left(\frac{1}{x_{k^*}}\right) < \sum_{i=2}^{n} \left(s_{i,k^*}\right) + \left(c_{k^*}\right) \leqslant 1 \tag{A.12}$$

which is contradictory to Equation (A.11). Otherwise, that is, if $c_{k^*}=0$, there exists a path in G from the root node to node k^*, $(1, k_1, \ldots, k_{l-1}, k_l = k^*)$, with $l \geqslant 2$. To satisfy Equation (A.11), $x_{k_{l-1}} = x_{k^*}$. By applying the same argument to $k_{l-1},\ldots,k_1$, we get $x_{k_l}=x_{k_{l-1}}=\cdots=x_{k_1}$. Equation (2) for x_{k_1} is written as

$$1=\sum_{i=2}^{n}\left(s_{i,k_1}\right)\left(\frac{x_i}{x_{k_1}}\right)+\left(c_{k_1}\right)\left(\frac{1}{x_{k_1}}\right) \tag{A.13}$$

Since $c_{k_1}>0$, we can see that the right-hand side of Equation (A.13) is less than 1, which is contradictory. Therefore, $x_{k*}\leqslant 1$. □

Proof of Theorem 2 Let $\mathbf{e}^j\in\Re^{n-1}$ denote a unit vector whose j-th component is 1 and other components are zero. By Sherman–Morrison formula, we have

$$\begin{aligned}\mathbf{x}_{\overline{N}}^{G'}&=\left(I-\overline{S}^T+s_{uv}\mathbf{e}^v(\mathbf{e}^u)^T\right)^{-1}\mathbf{c}\\&=\mathbf{x}_{\overline{N}}^{G}-\frac{s_{uv}x_u^G}{1+s_{uv}(\mathbf{e}^u)^T\left(I-\overline{S}^T\right)^{-1}\mathbf{e}^v}\left(I-\overline{S}^T\right)^{-1}\mathbf{e}^v\end{aligned}$$

Since $(I-\overline{S}^T)^{-1}=(I+\overline{S}^T+(\overline{S}^T)^2+\cdots)$, it is obvious that

$$\mathbf{x}^{G'}\leqslant\mathbf{x}^G$$

Since $(I-\overline{S}^T)$ is non-singular and $\left\|\left((I-\overline{S}^T)^{-1}\right)\right\|_2$ is equal to $\sigma_{\min}^{-1}$ (Golub and Van Loan, 1996), we obtain

$$\begin{aligned}\left\|\mathbf{x}^G-\mathbf{x}^{G'}\right\|_2&=\frac{s_{uv}x_u^G}{1+s_{uv}(\mathbf{e}^u)^T\left(I-\overline{S}^T\right)^{-1}\mathbf{e}^v}\left\|\left(I-\overline{S}^T\right)^{-1}\mathbf{e}^v\right\|_2\\&\leqslant s_{uv}x_u^G\left\|\left(I-\overline{S}^T\right)^{-1}\right\|_2\\&\leqslant\frac{s_{uv}x_u^G}{\sigma_{\min}}\end{aligned}$$ □

Proof of Theorem 3 If G has no cycle, there exists a topological ordering for G. According to the topological ordering, the voting right vector is uniquely determined. In addition, since $y_1=1$ and $\sum_{i\in Adj^{-1}(j)}s_{ij}\leqslant 1$, it is obvious that $0\leqslant y_i\leqslant 1$.

Next, suppose that G contains some cycles. Let $\mathcal{C}=(j_1,\ \ldots\ ,j_{r+1})$ be a cycle in G where $j_1=j_{r+1}$. Also, let $(j_l,j_{l+1})=(u,v)\in\mathcal{C}$ be an arc such that $s_{uv}=\min_{1\leqslant k\leqslant r}s_{j_k,j_{k+1}}$. Since $\min(s_{uv},y_u)=s_{uv}$ by Lemma 2 in the Appendix, we can replace $\min(y_u,s_{uv})$ in the system of Equation (3) with s_{uv}. Repeating the same procedure for all remaining cycles in G eventually results in the revised system of equations where y_j is no longer defined in a circular way. This implies that each y_j is uniquely determined in the revised systems of equations. Finally, let us show that $0\leqslant y_j\leqslant 1$. Since the nonnegativity of voting right vector $\mathbf{y}$ is guaranteed by the definition, it is sufficient to show that $y_j\leqslant 1$ for all j.

$$y_j=\sum_{i\in Adj^{-1}(j)}\min\left(s_{ij},y_i\right)\leqslant\sum_{i\in Adj^{-1}(j)}s_{ij}\leqslant 1$$ □

Lemma 2 *Let* $\mathbf{y}$ *denote the voting-right vector for an ownership network G. Also, let* $\mathcal{C}=(j=j_1,j_2,\ \ldots\ ,j_r,j=j_{r+1})$ *be a cycle contained in G. Let us assume without loss of generality that* $s_{j_1,j_2}=\min_{k=1,\ldots,r}s_{j_k,j_{k+1}}$. *Then,* $y_{j_1}\geqslant s_{j_1,j_2}$.

Proof Since $j_{k-1}\in Adj^{-}(j_k)$ for $k=2,\ldots,r+1$, each y_{j_k}, by definition, is determined by the following equation:

$$y_{j_k}=\min\left(s_{j_{k-1},j_k},y_{j_{k-1}}\right)+\sum_{i\in Adj^{-1}(j_k)-\{j_{k-1}\}}\min\left(s_{i,j_k},y_i\right) \tag{A.14}$$

First, we will show that $\mathcal{C}$ includes at least one node j_k such that $y_{j_k}\geqslant s_{j_k,j_{k+1}}$. Suppose $y_{j_k}<s_{j_k,j_{k+1}}$ for all $k=1,\ldots,r$. Then, Equation (A.14) implies that $y_{j_{k+1}}\geqslant y_{j_k}$. Applying the same argument to all arcs in the cycle, we obtain

$$y_{j_1}\leqslant y_{j_2}\leqslant\cdots\leqslant y_{j_r}\leqslant y_{j_{r+1}} \tag{A.15}$$

By Assumptions 1 and 2, cycle $\mathcal{C}$ includes at least one node $y_{j_{l+1}}$ that has some incoming arcs from a node that is not contained in $\mathcal{C}$. For $y_{j_{l+1}}$, we have

$$y_{j_l}<y_{j_{l+1}} \tag{A.16}$$

From Equations (A.15) and (A.16), we get

$$y_j=y_{j_1}<y_{j_{r+1}}=y_j \tag{A.17}$$

which is contradictory. Consequently, it holds that $y_{j_k}\geqslant s_{j_k,j_{k+1}}$ for some $j_k\in\mathcal{C}$ with $k=1,\ldots,r$.

Next, it will be shown that $y_{j_1}\geqslant s_{j_1,j_2}$. Suppose that $y_{j_k}\geqslant s_{j_k,j_{k+1}}$ for $j_k\in\mathcal{C}$ and $k\neq 1$. Then, we get

$$y_{j_{k+1}}\geqslant\min\left(y_{j_k},s_{j_k,j_{k+1}}\right)=s_{j_k,j_{k+1}} \tag{A.18}$$

If $k=r$, inequality (22) leads to

$$y_{j_{r+1}}\geqslant s_{j_r,j_{r+1}}\geqslant s_{j_1,j_2} \tag{A.19}$$

which establishes the lemma. Otherwise, we obtain

$$y_{j_{k+2}}\geqslant\min\left(y_{j_{k+1}},s_{j_{k+1},j_{k+2}}\right)\geqslant\min\left(s_{j_k,j_{k+1}},s_{j_{k+1},j_{k+2}}\right)\geqslant s_{j_1,j_2} \tag{A.20}$$

which can be repeatedly applied along the nodes in $\mathcal{C}$. Therefore, $y_{j_1}\geqslant s_{j_1,j_2}$. □

Lemma 3 *Suppose an ownership network, $G=(N,E)$, satisfies Assumptions 1 and 2. Let $S=(s_{ij})$ be the ownership matrix of G. Also, let $\overline{N}=\{2,\ \ldots\ ,n\}$ and $\overline{S}=S_{\overline{NN}}$ where node 1 is the root node. Then,*

(a) *if* $\mathbf{x}\leqslant\overline{S}^T\mathbf{x}$, *then* $\mathbf{x}\leqslant 0$

(b) *if* $\mathbf{x}\geqslant\overline{S}^T\mathbf{x}$, *then* $\mathbf{x}\geqslant 0$.

Proof Let $P=\{j|x_j>0\}$. To show (a), suppose that $\mathbf{x}\leqslant\overline{S}\mathbf{x}$ and $P\neq\emptyset$. Let $j^*=\operatorname{argmax}\{x_j|j\in P\}$. Then,

$$x_{j^*}\leqslant\left(\overline{S}^T\mathbf{x}\right)_{j^*}=\sum_{i=2}^{n}\left(s_{i,j^*}\right)(x_i)\leqslant\sum_{i\in P}\left(s_{i,j^*}\right)(x_i)$$
$$\leqslant\left(x_{j^*}\right)\sum_{i\in P}s_{i,j^*}. \tag{A.21}$$

Then, we have $\sum_{i\in P}s_{i,j^*}=1$. Otherwise, that is, if $\sum_{i\in P}s_{i,j^*}<1$, inequality (A.21) would be contradictory. If $x_k<x_{j^*}$ for some $k\in P$, then

$$x_{j^*}\leqslant\left(\overline{S}^T\mathbf{x}\right)_{j^*}\leqslant\sum_{i\in P}\left(s_{i,j^*}\right)(x_i)<\left(x_{j^*}\right)\sum_{i\in P}s_{i,j^*}$$
$$=x_{j^*} \tag{A.22}$$

which is contradictory. Hence, it must hold that $x_j=x_{j^*}$ for all $j\in P$. Since G has at least one path from the root node to any other node, there is at least one $q\in P$ such that $x_q=x_{j^*}$ and $\sum_{i\in P}s_{iq}<1$. By applying the same arithmetic of inequality (A.21) to q instead of j^*, we can reach a contradiction. Therefore, P should be an empty set.

Next, let us show (b) of the lemma. From (a) of the lemma, we derive

$$\mathbf{x}\leqslant\overline{S}^T\mathbf{x}\Leftrightarrow(-\mathbf{x})\geqslant\overline{S}^T(-\mathbf{x}) \tag{A.23}$$

which establishes (b) of the lemma. □

Proof of Theorem 4 For $\mathbf{u}\in\Re^{1\times q}$ and $\mathbf{v}\in\Re^{q\times 1}$ with $q\geqslant 1$, let us define operator $<\cdot,\cdot>$ as the following:

$$<\mathbf{u},\mathbf{v}>=\min(u_1,v_1)+\cdots+\min\left(u_q,v_q\right)$$

Without any ambiguity, the operator $<\cdot,\cdot>$ can be extended for $U\in\Re^{p\times q}$ and $v\in\Re^{p\times 1}$ as the following:

$$<U,\mathbf{v}>=\begin{bmatrix}<U_{1\cdot},\mathbf{v}>\\ \vdots\\ <U_{p\cdot},\mathbf{v}>\end{bmatrix}$$

where $U_{i\cdot}$ denotes the i-th row vector of U. For notational convenience, let $\overline{\mathbf{y}}=\mathbf{y}^G_{\overline{N}}$ and $\overline{\mathbf{x}}=\mathbf{x}^G_{\overline{N}}$. By definition, $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ satisfy the following equations:

$$\overline{\mathbf{y}}=\mathbf{c}+<\overline{S}^T,\overline{\mathbf{y}}> \tag{A.24}$$

$$\overline{\mathbf{x}}=\mathbf{c}+\overline{S}^T\overline{\mathbf{x}} \tag{A.25}$$

where $\mathbf{c}=(s_{12},\ldots,s_{1n})^T$.

Let $B=\{j\in\overline{N}\ \mid\ \overline{x}_j>\overline{y}_j\}$ and $D=\overline{N}-B$. The theorem is equivalent to a claim that B is an empty set. For contradiction, suppose that $B\neq\emptyset$. Since $0\leqslant s_{ij}\leqslant 1$ and $\min(s_{ij},\beta)\in s_{ij}\ \beta$ for any $0\leqslant\beta\leqslant 1$, it holds that $<\overline{S}^T,\overline{\mathbf{y}}>\ \geqslant\ \overline{S}^T\mathbf{y}$. Taking the inner product between $\overline{\mathbf{x}}$ and Equation (A.24) and between $\overline{\mathbf{y}}$ and Equation (A.25), we obtain

$$\overline{\mathbf{y}}^T\overline{\mathbf{x}}=\mathbf{c}^T\overline{\mathbf{y}}+\overline{\mathbf{y}}^T\overline{S}^T\overline{\mathbf{x}}=\mathbf{c}^T\overline{\mathbf{x}}+\overline{\mathbf{x}}^T\left(<\overline{S}^T,\overline{\mathbf{y}}>\right)$$
$$\geqslant\mathbf{c}^T\overline{\mathbf{x}}+\overline{\mathbf{x}}^T\overline{S}^T\overline{\mathbf{y}} \tag{A.26}$$

Hence, $\mathbf{c}^T\overline{\mathbf{y}}\geqslant\mathbf{c}^T\overline{\mathbf{x}}$, which indicates that D is not an empty set if $B\neq\emptyset$. Equation (A.25) can be rewritten as

$$\begin{bmatrix}\overline{\mathbf{x}}_B\\ \overline{\mathbf{x}}_D\end{bmatrix}=\begin{bmatrix}\mathbf{c}_B\\ \mathbf{c}_D\end{bmatrix}+\begin{bmatrix}\overline{S}^T_{BB} & \overline{S}^T_{DB}\\ \overline{S}^T_{BD} & \overline{S}^T_{DD}\end{bmatrix}\begin{bmatrix}\overline{\mathbf{x}}_B\\ \overline{\mathbf{x}}_D\end{bmatrix}$$
$$=\begin{bmatrix}\left(\mathbf{c}_B+\overline{S}^T_{DB}\overline{\mathbf{x}}_D\right)+\overline{S}^T_{BB}\overline{\mathbf{x}}_B\\ \left(\mathbf{c}_D+\overline{S}^T_{DD}\overline{\mathbf{x}}_D\right)+\overline{S}^T_{BD}\overline{\mathbf{x}}_B\end{bmatrix} \tag{A.27}$$

Let $\tilde{\mathbf{x}}_B$ be a vector such that

$$\tilde{\mathbf{x}}_B=\left(\mathbf{c}_B+\overline{S}^T_{DB}\overline{\mathbf{y}}_D\right)+\overline{S}^T_{BB}\tilde{\mathbf{x}}_B \tag{A.28}$$

Note that by Lemma 1, Equation (A.28) has one unique solution. From Equations (A.27) and (A.28), we obtain

$$\tilde{\mathbf{x}}_B-\overline{\mathbf{x}}_B=\overline{S}^T_{DB}(\overline{\mathbf{y}}_D-\overline{\mathbf{x}}_D)+\overline{S}^T_{BB}(\tilde{\mathbf{x}}_B-\overline{\mathbf{x}}_B)\geqslant\overline{S}^T_{BB}(\tilde{\mathbf{x}}_B-\overline{\mathbf{x}}_B) \tag{A.29}$$

By Lemma 3 in the Appendix, inequality (33) indicates that $\tilde{\mathbf{x}}_B\geqslant\overline{\mathbf{x}}_B$. Also, from Equation (A.24), we obtain

$$\overline{\mathbf{y}}=\mathbf{c}+<\overline{S}^T,\overline{\mathbf{y}}>\geqslant\mathbf{c}+\overline{S}^T\overline{\mathbf{y}} \tag{A.30}$$

Combining Equation (A.28) and inequality (34), we get

$$\overline{\mathbf{y}}_B-\tilde{\mathbf{x}}_B\geqslant\overline{S}^T_{BB}(\overline{\mathbf{y}}_B-\tilde{\mathbf{x}}_B) \tag{A.31}$$

By Lemma 3, inequality (35) indicates that $\overline{\mathbf{y}}_B\geqslant\tilde{\mathbf{x}}_B\geqslant\overline{\mathbf{x}}_B$, which contradicts that $B\neq\emptyset$. □

Received 24 January 2013;
accepted 21 October 2014 after two revision